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Derived category methods in commutative algebra. (English) Zbl 1559.13001
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Homological algebra is a mathematical framework that initially emerged from classical simplicial topology, essentially as a crystallization product of the concepts of homology and cohomology groups of topological spaces. In the early 1940's Eilenberg and MacLane started to formalize the concept of (co-)homology to an abstract concept of "homological" methods. A first systematic approach to homological algebra was the epoch-making book by *H. Cartan* and *S. Eilenberg* [Homological algebra. Princeton, NJ: Princeton University Press(1956; Zbl 0075.24305)]. Their book has a strong influence by making the topic more accessible to a broader community of Mathematicians. Moreover, E. Noether with her pioneering work opens the field of Commutative Algebra. The subject was further developed by the strong influence of W. Krull, O. Zariski, P. Samuel and M. Nagata. A central question at that time was: Is the localization of a regular local ring again regular? As it was known by the regular points of an algebraic variety. This problem was solved affirmatively in general by *J.-P. Serre* [in: Proc. Int. Symp. Algebraic Number Theory 1955, 175–189 (1956; Zbl 0073.26004); *J.-P. Serre* and *P. Gabriel* (ed.), Algèbre locale. Multiplicités. (Local algebra. Multiplicities). Cours au Collège de France, 1957-1958. Rédigé par Pierre Gabriel. Troisième éd., 2nd corrected printing. Berlin etc.: Springer-Verlag (1989; Zbl 0661.13008)] and by *M. Auslander* and *D. A. Buchsbaum* [Ann. Math. (2) 68, 625–657 (1958; Zbl 0092.03902)]. It was proved by a homological characterization of regular local rings as those with finite global dimension. This and related results (e.g. that a regular ring is factorial) opens the view towards the study of problems in Commutative Algebra by homological methods (see also the study of Gorenstein rings by *H. Bass* [Math. Z. 82, 8–28 (1963; Zbl 0112.26604)].

These results initiated the investigations of resolutions by several types of modules and more general the investigations of complexes of modules and extending numerical data of modules to complexes. One starting point of those investigations was the work of *L. L. Avramov* and *H.-B. Foxby* [J. Pure Appl. Algebra 71, No. 2–3, 129–155 (1991; Zbl 0737.16002)]. While isomorphisms are the right frame for the work with modules this is not the case for the study of complexes because mostly interested in its (co-)homology. To this end A. Grothendieck suggested to consider the category whose objects are complexes and whose morphisms are homotopy equivalence classes of morphisms of complexes. The derived category is obtained by "localizing" this category so that every morphism which induces an isomorphism in cohomology becomes an isomorphism in the derived category. This was outlined by *R. Hartshorne* [Residues and duality. Lecture notes of a seminar on the work of A. Grothendieck, given at Havard 1963/64. Appendix: Cohomology with supports and the construction of the $f^!$ functor by P. Deligne. Springer, Cham (1966; Zbl 0212.26101)] in order to prove a duality theorem for cohomology of quasi-coherent sheafs with respect to a proper morphism of local Noetherian preschemes and by *J.-L. Verdier* in his thesis [Lect. Notes Math. 569, 262–311 (1977; Zbl 0407.18008)]. First textbook representations of derived categories and derived functors one can find by *S. I. Gel'fand* and *Yu. I. Manin* [Методы гомологической алгебры. Том 1: Введение в теорию когомологий и производные категории (Russian). Moskva: Nauka (1988; Zbl 0668.18001)] and *C. A. Weibel* [An introduction to homological algebra. Cambridge: Cambridge University Press (1994; Zbl 0797.18001)].

By view of the joint papers of Avramov and Foxby (see [loc. cit.]) Foxby started with "A homological theory of complexes of modules" (see [Preprint Series 1981 nos. 19 a & b, Matematisk Institut, Københavns Universitet (1981)]). In the Introduction he wrote: These notes are an attempt to support the above assertion ("it is often beneficial to replace "modules" by "complexes of modules") by showing how nicely one part of the homological theory of modules extends to a homological theory of bounded complexes of modules. In their paper [Adv. Math. 133, No. 1, 54–95 (1998; Zbl 0935.13008)] *L. L. Avramov* and *H.-B. Foxby* applied derived category methods to study a wider class of ring homomorphisms. This leads to several drafts about "Hyperhomological algebra & commutative rings" by H. B. Foxby and the first author of this book. The present book is the outcome of these drafts (as the authors write): "The goal of this book is to expound the utility of derived category methods in ring theory. For commutative rings, the goal is further to collect textbook applications of these methods into a compact, yet comprehensive, introduction to homological aspects of commutative algebra."

The voluminous outcome is the present book of 1119 pages with 299 references and 25 pages of Index. The authors assign their work as a graduate textbook and a work of reference as well as a basis for teaching Homological Algebra with a view towards Commutative Algebra. To this end they included to each section exercises in order to perform elementary verifications and computations that are omitted from the text. The book is devided into three parts (I Foundations, II Tools, III Applications – in Commutative Algebra) and Appendices.

In the first part I Foundations (divided into 6 sections) the authors provide an introduction to the basics of homological algebra over a commutative ring. This culminates in the extension of freeness, projectivity, injectivity, flatness of the case of modules used for the corresponding resolutions to the situation of unbounded complexes and their resolutions. Here they do not follow the technique by *N. Spaltenstein* [Compos. Math. 65, No. 2, 121–154 (1988; Zbl 0636.18006)] but the different approach developed by *L. Avramov* et al. (see ["Differential graded homological algebra", Preprint (1994–2017)]) based on a technique colloquially known to as "killing cycles". That is, the existence of semi-free, semi-projective, semi-injective, semi-flat resolutions of an arbitary complex is proved. In the final chapter of this section they construct the derived category of the module category. That becomes an effective environment for the homological study of modules as shown in the subsequent parts of the book.

Part II Tools is devoted to the construction of derived functors, in particular derived Hom and derived tensor product with the standard isomorphisms between them. Together with the standard isomorphisms the evaluation morphisms play a central role, in particular for their applications in the third part of the book. With this in mind there is a chapter about homological dimensions (i.e. projective, injective, flat) of complexes and global and finitistic dimension. A "refinement" of these classical homological dimensions is treated in a separate chapter "Gorenstein homological dimensions", originated by the work of *M. Auslander* and *M. Bridger* [Stable module theory. Providence, RI: American Mathematical Society (AMS) (1969; Zbl 0204.36402)]. The approach here follows the work by L. Avramov, H. B. Foxby, L. W. Christensen, A. J. Frankhild, H. Holm (see the first author's Lecture note [Gorenstein dimensions. Berlin: Springer (2000; Zbl 0965.13010)] and the references there). Part II is completed by a Chapter on Grothendieck Duality and Dualizing Complexes. A final chapter completion, torsion, local (co-)homology and their relation to Koszul and Čech complexes is added.

Part III "Applications – in commutative algebra" covers almost half of the book. It follows the homological theory of commutative Noetherian rings. It starts with a Chapter that recapitulates principal results of Parts I and II that simplify in the case of commutative rings. In the Chapter "Derived torsion and completion" the Čech local (co-)homologies are studied. For further results in the case of pro-regular sequences see also [*P. Schenzel* and *A.-M. Simon*, Completion, Čech and local homology and cohomology. Interactions between them. Cham: Springer (2018; Zbl 1402.13001)]. Moreover, there is an extension of notions from commutative algebra (support, Krull dimension, depth, width) to complexes and the description of depth and width by local (co-)homology. The classical notion of support and cosupport (invented by *L. Melkersson* and *P. Schenzel* [Proc. Edinb. Math. Soc., II. Ser. 38, No. 1, 121–131 (1995; Zbl 0824.13011)] and independently by *M. Hovey* and *N. P. Strickland* [Morava K-theories and localisation. Providence, RI: American Mathematical Society (AMS) (1999; Zbl 0929.55010)]) are extended to complexes and are shown to be more relevant in the derived category setting than classic notion of modules. In the Chapter "Homological invariants over local rings" they show the "classical" results about homological invariants of local rings with a certain view towards complexes. They use it for the study of homological invariants over general rings through the local-global principle. Chapter 18 contains several of the classic big results in the field of Commutative Algebra like Matlis and Grothendieck Duality Theorems, the Local Duality Theorem, the New Intersection Theorem, and the related homological characterizations of Cohen-Macaulay rings. Several results of the homological theory over commutative Noetherian rings developed in the book have counterparts in the theory of Gorenstein homological dimensions developed in Chapter 19. In the final Chapter 20 the authors characterize regular local rings as those of global finite dimension (Auslander-Buchsbaum, Serre Theorems) and prove Serre's Intersection Theorem (see [loc. cit.]) and Iversen's amplitude inequality – an extension of Serre's result to complexes (see [*B. Iversen*, Ann. Sci. Éc. Norm. Supér. (4) 10, 547–558 (1977; Zbl 0433.13003)]). They discuss additional classes of local rings (complete intersection, Corenstein, Cohen-Macaulay) and compare them as well as non-local regular rings. Constructions and results that are needed in order to make the book self-contained and are a bit outside of the flow of the presentation are collected in Appendices (among them "Structure of injective modules", "Projective dimension of flat modules").

As an outcome the book indicates that the derived category methods have significant advantages over the classical homological algebra in commutative algebra and extend these methods. For the proofs of the results about the homological conjectures of (M. Hochster, P. Roberts, Y. André) the authors refer to the original papers. While the authors claim the material of this textbook is not new it provides a new and conclusive presentation of the homological methods, in particular the use of derived category foundations and their application. Surely it will be well-come to graduate students and researchers for a reference as well as a basis for further research. Also it might be used for graduate introductory as well as advanced courses. It offers a more detailed presentation of the subject than in the most existing textbooks. For the reviewer's opinion the book is a very useful, up-to-date reference book for working mathematicians in Commutative Algebra using homological methods.

Reviewer: Peter Schenzel (Halle an der Saale)

MSC:

13-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to commutative algebra
13Dxx Homological methods in commutative ring theory
16-00 General reference works (handbooks, dictionaries, bibliographies, etc.) pertaining to associative rings and algebras
16Exx Homological methods in associative algebras
18-00 General reference works (handbooks, dictionaries, bibliographies, etc.) pertaining to category theory
18Gxx Homological algebra in category theory, derived categories and functors

Keywords:

derived category; derived functor; homological dimension and commutative rings; homological functors; injective, projective, flat resolutions of complexes; special types of commutative rings

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